

Nonfiliform characteristically nilpotent Lie algebras*

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Abstract

In this paper we construct large families of characteristically nilpotent Lie algebras by considering deformations of the Lie algebra $\mathfrak{g}_{(m,m-1)}^4$ of type Q_n , and which arises as a central naturally graded extension of the filiform Lie algebra L_n . By studying the graded cohomology spaces we obtain that the nil algebras associated to the models $\mathfrak{g}_{(m,m-1)}^4$ can be interpreted as nilradicals of solvable, complete Lie algebras. For extreme cocycles we obtain moreover nilradicals of rigid laws. By considering supplementary cocycles we construct, for any dimension $n \geq 9$, non-filiform characteristically nilpotent Lie algebras and show that for certain deformations these deformations are compatible with central extensions.

Introduction

Characteristically nilpotent Lie algebras were born in the late 50's, as the answer to a question formulated in 1955 by Jacobson. In [14] he proved that any Lie algebra defined over a field of characteristic zero and admitting nondegenerate derivations is nilpotent, and asked for the converse. Dixmier and Lister gave a negative answer in 1957, constructing an eight dimensional Lie algebra all whose derivations are nilpotent. The so called characteristically nilpotent Lie algebras constitute an important subclass within the nilpotent algebras, and they have been studied in a number of papers ([15], [19])

Over the last years the study of characteristically nilpotent Lie algebras has been reactivated by the fact that they are of great importance for the topological analysis of the irreducible components of the variety of nilpotent Lie algebra laws \mathfrak{N}^n [27]. Though there are a lot of constructions of such algebras, most results correspond to filiform Lie algebras, i. e., algebras with maximal nilpotence

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index. In [17] the author studied the graded filiform Lie algebra L_n profoundly, and deduced interesting results about the topology of its components, as well as giving large families of characteristically nilpotent Lie algebras.

In this paper we study deformations of a family $\mathfrak{g}_{(m,m-1)}^4$ of graded nilpotent Lie algebras of characteristic sequence $(n-2, 1, 1)$, and which are k -abelian of maximal index $k = \lfloor \frac{n-2}{2} \rfloor$. These algebras are the simplest case of a wide class of graded Lie algebras called "of type Q_n ", and which have been introduced in [4]. The family we are considering is of special interest, as it is the only algebra of type Q_n which is a central, naturally graded extension of the filiform Lie algebra L_n . The characteristically nilpotent algebras obtained by deformations of this model are of value for the general theory, because it is one of the few (if any) explicit methods to obtain non-filiform characteristically nilpotent Lie algebras (see also [3]).

The paper is divided in four parts. In the first section we recall the elementary facts about the invariants of nilpotent Lie algebras used, as well as the rudiments of characteristically nilpotent Lie algebras. In section two we study the cohomology of nilpotent Lie algebras and introduce a certain partition of the cohomology groups $H^2(\mathfrak{g}, \mathbb{C})$. In the third part we analyze the model $\mathfrak{g}_{(m,m-1)}^4$ and its deformations to obtain, on one hand, deformations which are isomorphic to the nilradical of solvable, complete laws, which are rigid for certain cocycles, and on the other, deformations which are characteristically nilpotent. Finally, we prove that certain prolongations of these cocycles are compatible with central extensions of degree one of $\mathfrak{g}_{(m,m-1)}^4$ and construct characteristically nilpotent Lie algebras with characteristic sequence $(2m-1, 2, 1)$ for any $m \geq 4$.

1 Generalities

Let \mathfrak{L}^n be the set of complex Lie algebra laws in dimension n . We identify each law with its structure constants C_{ij}^k on a fixed basis $\{X_i\}$ of \mathbb{C}^n . The Jacobi identities

$$\sum_{l=1}^n C_{ij}^k C_{kl}^s + C_{jk}^l C_{il}^s + C_{ki}^l C_{jl}^s = 0$$

for $1 \leq i \leq j < k \leq n$, $1 \leq s \leq n$ show that \mathfrak{L}^n is an algebraic variety. The nilpotent Lie algebra laws \mathfrak{N}^n are a closed subset in \mathfrak{L}^n . The linear group $GL(n, \mathbb{C})$ acts on \mathfrak{L}^n by changes of basis. If $\mathcal{O}(\mu)$ denotes the orbit of the law μ by this action, it is easy to see that it is a regular subvariety [12].

Definition. *If the orbit $\mathcal{O}(\mu)$ is open in \mathfrak{L}^n (resp. \mathfrak{N}^n), the law μ is rigid in \mathfrak{L}^n (resp. \mathfrak{N}^n).*

For later use, it is convenient to recall some notation :

$$N_p^n = \{\mu \in \mathfrak{N}^n \mid \text{nilindex}(\mu) \leq p-1\}$$

1.1

Consider a complex nilpotent Lie algebra $\mathfrak{g} = (\mathbb{C}^n, \mu)$. For each $X \in \mathbb{C}^n$ we denote $c(X)$ the ordered sequence of dimensions of Jordan blocks of the adjoint operator $ad_\mu(X)$.

Definition. The characteristic sequence of \mathfrak{g} is an isomorphism invariant $c(\mathfrak{g})$ defined as

$$c(\mathfrak{g}) = \sup_{X \in \mathfrak{g} - C^1 \mathfrak{g}} \{c(X)\}$$

where $C^1 \mathfrak{g}$ denotes the derived subalgebra.

Definition. An n -dimensional nilpotent Lie algebra \mathfrak{g} is called filiform if $c(\mathfrak{g}) = (n-1, 1)$.

Recall that the filiform model algebras L_n and Q_n are :

1. L_n is the $(n+1)$ -dimensional Lie algebra ($n \geq 3$) defined by

$$[X_1, X_i] = X_{i+1}, \quad 2 \leq i \leq n$$

where $\{X_1, \dots, X_{n+1}\}$ is a basis.

2. Q_{2m-1} is the $2m$ -dimensional Lie algebra ($m \geq 3$) defined by

$$\begin{aligned} [X_1, X_i] &= X_{i+1}, & 2 \leq i \leq 2m-1 \\ [X_j, X_{2m+1-j}] &= (-1)^j X_{2m}, & 2 \leq j \leq m \end{aligned}$$

where $\{X_1, \dots, X_{2m}\}$ is a basis.

1.2

Definition. Let \mathfrak{g} be a solvable, non nilpotent Lie algebra. Then \mathfrak{g} is called decomposable if

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t}$$

where \mathfrak{n} is the nilradical of \mathfrak{g} and \mathfrak{t} an exterior torus of derivations, i.e., an abelian subalgebra consisting of ad-semisimple endomorphisms.

The structure of these algebras is well known by a theorem of Carles [6] :

Theorem. A rigid solvable Lie algebra \mathfrak{g} is decomposable.

Definition. If \mathfrak{t} is a maximal torus for the Lie algebra \mathfrak{g} , then $r = \dim \mathfrak{t}$ is called the rank of \mathfrak{g} , noted $\text{rank}(\mathfrak{g})$.

1.3

We recall here the elementary facts about characteristically nilpotent Lie algebras.

Definition. A Lie algebra \mathfrak{g} is called characteristically nilpotent if the algebra of derivations $\text{Der}(\mathfrak{g})$ is nilpotent.

From this definition it follows that \mathfrak{g} is itself nilpotent.

Proposition. The characteristically nilpotent Lie algebras constitute a constructible set in the variety \mathfrak{N}^n , empty for $n \leq 6$ and nonempty for $n \geq 7$.

A proof of this result can be found in [7].

2 Cohomology of nilpotent Lie algebras

Let \mathfrak{g} be a nilpotent Lie algebra whose nilindex is p . We have the ascending and descending central sequences $\{C^j \mathfrak{g}\}$ and $\{C_j \mathfrak{g}\}$, which are respectively defined by

$$C^0 \mathfrak{g} = \mathfrak{g}, C^q \mathfrak{g} = [C^{q-1}(\mathfrak{g}), \mathfrak{g}], \text{ for } q \geq 1$$

$$C_0 \mathfrak{g} = 0, C_q \mathfrak{g} = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset C_{q-1}(\mathfrak{g})\}$$

We set

$$S_q = \mathfrak{g}, \text{ for } q \leq 1, S_q = C^{q-1}(\mathfrak{g}), q \geq 1$$

$$T_q = \mathfrak{g}, \text{ for } q \leq 1, T_q = C_{p+1-q}(\mathfrak{g}), q \geq 1$$

obtaining two descending filtrations of \mathfrak{g} . It is usual to consider the algebra \mathfrak{g} filtered by $\{S_q\}$ and the \mathfrak{g} -module \mathfrak{g} (respect to the adjoint representation) filtered by $\{T_q\}$. This choice induces descending filtrations

$$\{F_k Z^j(\mathfrak{g}, \mathfrak{g})\}, \{F_k H^j(\mathfrak{g}, \mathfrak{g})\}, \{F_k B^j(\mathfrak{g}, \mathfrak{g})\}$$

in the cohomology space, which is compatible with the coboundary operator. The previous filtration has been profoundly studied by Khakimdjanoj for the filiform Lie algebra L_n in [15], where he obtained important results on the topology of the irreducible components of the variety \mathfrak{N}^n of nilpotent Lie algebra laws.

Let

$$d_i = \dim_{\mathbb{C}} S_i, 1 \leq i \leq p$$

$$d_{p+1} = 0$$

Lemma. .

- 1) If $j \in \mathbb{N}$ and $j > d_1$, then $F_r Z^j(\mathfrak{g}, \mathfrak{g}) = Z^j(\mathfrak{g}, \mathfrak{g}) = 0$, $r \in \mathbb{Z}$
- 2) If $d_s < j < d_{s+1}$ for some $1 \leq s \leq p$, then $F_r Z^j(\mathfrak{g}, \mathfrak{g}) = Z^j(\mathfrak{g}, \mathfrak{g})$ for $r \leq q$, where

$$q = -[pd_p + (p-1)(d_{p-1} - d_p) + \dots + s(d_s - d_{s+1}) + (s-1)(j-1-d_s)]$$

A proof can be found in [16].

Lemma. If $r \leq p(1-j)$ then $F_r Z^j(\mathfrak{g}, \mathfrak{g}) = Z^j(\mathfrak{g}, \mathfrak{g})$.

Corollary. If f is a derivation of \mathfrak{g} , then $f(S_i) \subset T_i$ for any i .

These results, due originally to Vergne [26], were used by Khakimdjanoj [16] to construct large families of characteristically nilpotent Lie algebras which are deformations of the filiform model algebra L_n .

2.1

Recall that a central extension of a Lie algebra \mathfrak{g} by \mathbb{C}^p is an exact sequence of Lie algebras

$$0 \rightarrow \mathbb{C}^p \rightarrow \bar{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$$

with $\mathbb{C}^p \subset Z(\bar{\mathfrak{g}})$. If μ_0 is the law of $\mathfrak{g} = (C^m, \mu_0)$, the law of $\bar{\mathfrak{g}}$ can be expressed as

$$\mu(\varphi)(X, Y) = \begin{cases} \mu_0(X, Y) + \varphi(X, Y), & (X, Y) \in \mathfrak{g}^2 \\ 0, & (X, Y) \in \mathbb{C}^p \times \mathbb{C}^m \end{cases}$$

where $\varphi \in Z^2(\mathfrak{g}, \mathbb{C}^p)$. The orbit $\mu(Z^2(\mathfrak{g}, \mathbb{C}^p))$ by the action of the group $GL(m+p, \mathbb{C})$ consists of the law over \mathbb{C}^{m+p} obtained by this manner.

The following result, due to Carles [6], is of importance in the study of characteristically nilpotent Lie algebras :

Proposition. The subspace $E_{c,p}(\mathfrak{g})$ of laws in \mathfrak{N}^{m+p} which are (up to isomorphism) central extensions of \mathfrak{g} by \mathbb{C}^p is constructible, irreducible and of dimension

$$m(m+p) - \dim \text{Der}(\mathfrak{g}) + p \dim Z^2(\mathfrak{g}, \mathbb{C}^p) - \rho$$

where ρ is the minimum for $\bar{\mathfrak{g}} \in E_{c,p}(\mathfrak{g})$ of the dimensions of subspaces of the Grassmannian $Gr_p(Z(\bar{\mathfrak{g}}))$ consisting of ideals $\mathfrak{a} \subset Z(\bar{\mathfrak{g}})$ such that $\frac{\bar{\mathfrak{g}}}{\mathfrak{a}} \simeq \mathfrak{g}$.

Proof. The space $E_{c,p}(\mathfrak{g})$ is the image of $Gl(m+p, \mathbb{C}) \times Z^2(\mathfrak{g}, \mathbb{C}^p)$ by the algebraic map

$$(s, \varphi) \mapsto s * \mu(\varphi)$$

where $*$ denotes the action of $Gl(m+p, \mathbb{C})$ on \mathfrak{N}^{m+p} . It follows that $E_{c,p}(\mathfrak{g})$ is irreducible and constructible. If ϕ is the law of $\bar{\mathfrak{g}}$, then its inverse image is algebraically isomorphic to the first projection

$$X(\phi) = \{s \in GL(m+p, \mathbb{C}) \mid s^{-1} * \phi \in \mu(Z^2(\mathfrak{g}, \mathbb{C}^p))\}$$

The algebraic mapping which associates $s(\mathbb{C}^p)$ to $s \in X(\phi)$ in the Grassmannian $Gr_p(\mathbb{C})$ by the image of the set of central ideals of $\bar{\mathfrak{g}}$ whose factor is isomorphic to \mathfrak{g} . If s_1 and s_2 give the same image \mathfrak{h} then $s_1^{-1}s_2$ belongs to the stabilizer of \mathbb{C}^p , mapping $\mu(\varphi_2)$ on $\mu(\varphi_1)$. We deduce that the minimum of dimensions for $X(\phi)$ equals $\dim Der(\mathfrak{g}) + mp + \rho$. \square

As known, the second cohomology space with values in \mathbb{C} is obtained as follows : Let $\lambda : \bigwedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ be the linear mapping defined by

$$\lambda(X \wedge Y) = \mu_0(X, Y), \quad X, Y \in \mathfrak{g}$$

and let Ω be the vector subspace of $\bigwedge^2 \mathfrak{g}$ generated by the elements

$$[X, Y] \wedge Z + [Y, Z] \wedge X + [Z, X] \wedge Y, \quad (X, Y, Z) \in \mathfrak{g}^3$$

It follows that the second homology space $H_2(\mathfrak{g}, \mathbb{C})$ coincides with the factor $\frac{Ker \lambda}{\Omega}$, and by duality

$$H^2(\mathfrak{g}, \mathbb{C}) = Hom\left(\frac{Ker \lambda}{\Omega}, \mathbb{C}\right) = H_2(\mathfrak{g}, \mathbb{C})^*$$

This space is of special interest in the determination of central extensions of degree 1. Now, let $\{X_1, \dots, X_n\}$ be a basis for the Lie algebra \mathfrak{g} and define the cocycles $\varphi_{ij} \in H^2(\mathfrak{g}, \mathbb{C})$, $(i, j, k, l \leq n)$

$$\varphi_{ij}(X_k, X_l) = \begin{cases} \delta_{ij} \in \mathbb{C} & \text{if } i = k, j = l \\ 0 & \text{otherwise} \end{cases}$$

where δ_{ij} denotes the Kronecker delta function. It is easy to verify that $\sum_{i,j} a^{ij} \varphi_{ij} = 0$ ($a_{ij} \in \mathbb{C}$) if and only if $\sum a^{ij} (X_i \wedge X_j) \in \Omega$.

Notation. For $k \geq 2$ let

$$H_k^{2,t}(\mathfrak{g}, \mathbb{C}) = \{\varphi_{ij} \in H^2(\mathfrak{g}, \mathbb{C}) \mid i+j = 2t+1+k, 1 \leq t \leq \left\lfloor \frac{\dim \mathfrak{g} - 3}{2} \right\rfloor\}$$

$$H_k^{2, \frac{t}{2}}(\mathfrak{g}, \mathbb{C}) = \{\varphi_{ij} \in H^2(\mathfrak{g}, \mathbb{C}) \mid i+j = t+1+k, 1 \leq t \leq \left\lfloor \frac{\dim \mathfrak{g} - 3}{2} \right\rfloor, t \equiv 1 \pmod{2}\}$$

If $\mathbf{E}_{c,1}(\mathfrak{g})$ defines the subset of $E_{c,1}(\mathfrak{g})$ formed by the central extensions of \mathfrak{g} by \mathbb{C} which are additionally naturally graded, we can define the sets

$$\mathbf{E}_{c,1}^{t,k_1,\dots,k_r}(\mathfrak{g}) = \left\{ \mu \in E_{c,1}(\mathfrak{g}) \mid \mu = \mu_0 + \left(\sum a_{kij} \varphi_{ij}^k \right), \varphi_{ij} \in H_k^{2,t}(\mathfrak{g}, \mathbb{C}), a_{kij} \in \mathbb{C} \right\}$$

$$\mathbf{E}_{c,1}^{\frac{t}{2},k_1,\dots,k_r}(\mathfrak{g}) = \left\{ \mu \in E_{c,1}(\mathfrak{g}) \mid \mu = \mu_0 + \left(\sum a_{kij} \varphi_{ij}^k \right), \varphi_{ij} \in H_k^{2,\frac{t}{2}}(\mathfrak{g}, \mathbb{C}), a_{kij} \in \mathbb{C} \right\}$$

If $\{X_1, \dots, X_n, X_{n+1}\}$ is a basis of $\bar{\mathfrak{g}} \in E_{c,1}(\mathfrak{g})$ the its law μ is given by :

$$\mu(X, Y) := \mu_0(X, Y) + \left(\sum a_{kij} \varphi_{ij}^k \right) X_n, \quad (X, Y) \in \mathfrak{g}^2$$

Lemma. *The following set-theoretically identity holds*

$$\mathbf{E}_{c,1}(\mathfrak{g}) = \bigsqcup_{t,k_i} \left(\mathbf{E}_{c,1}^{t,k_1,\dots,k_r}(\mathfrak{g}) \cup \mathbf{E}_{c,1}^{\frac{t}{2},k_1,\dots,k_r}(\mathfrak{g}) \right)$$

The proof is trivial.

Example. *For the Lie algebra \mathfrak{g}_7 defined by the brackets*

$$\begin{aligned} [X_1, X_i] &= X_{i+1}, \quad 2 \leq i \leq 5 \\ [X_2, X_3] &= X_7 \end{aligned}$$

we have the decomposition

$$\mathbf{E}_{c,1}(\mathfrak{g}_7) = \mathbf{E}_{c,1}^{\frac{5}{2},1}(\mathfrak{g}_7) \cup \mathbf{E}_{c,1}^{\frac{3}{2},3}(\mathfrak{g}_7) \cup \mathbf{E}_{c,1}^{\frac{3}{2},4}(\mathfrak{g}_7) \cup \mathbf{E}_{c,1}^{\frac{3}{2},4,5}(\mathfrak{g}_7)$$

Remark. *Recall that the 2-cocycles $\psi \in Z^2(\mathfrak{g}, \mathfrak{g})$ of the Chevalley cohomology can be interpreted as the infinitesimal deformations of the Lie algebra \mathfrak{g} .*

Definition. *Let $\psi \in Z^2(\mathfrak{g}, \mathfrak{g})$ be a cocycle of the Lie algebra $\mathfrak{g} = (\mathbb{C}^n, \mu)$. Then ψ is called linearly expandable if the operation*

$$(\mu + \psi)(X, Y) = \mu(X, Y) + \psi(X, Y)$$

defines a Lie algebra law over \mathbb{C}^n .

3 The algebra $\mathfrak{g}_{(m,m-1)}^4$

In this section we are interested on a certain class of nilpotent Lie algebras, which are called "Lie algebras of type Q_n ".

Definition. *A nilpotent Lie algebra \mathfrak{g} is called k -abelian if the ideal $C^k \mathfrak{g}$ of the central descending sequence is abelian and $C^{k-1} \mathfrak{g}$ is not abelian. The index k is called abelianity index of the Lie algebra \mathfrak{g} .*

Remark. If p is the nilindex of \mathfrak{g} , it is trivial to verify that

$$k \leq \left\lfloor \frac{p}{2} \right\rfloor$$

where $\lfloor (\cdot) \rfloor$ is the integer part function.

Definition. A k -abelian nilpotent Lie algebra of nilindex p is called of type Q_n if $k = \left\lfloor \frac{p}{2} \right\rfloor$.

The denomination is inspired on the following property of the filiform algebra Q_n :

Proposition. Let \mathfrak{g} be a naturally graded, n -dimensional filiform Lie algebra. Then \mathfrak{g} is k -abelian of maximal abelianity index if and only

- 1) $\dim \mathfrak{g} = 2m$
- 2) $\mathfrak{g} \simeq Q_{2m-1}$.

Remark. Vergne proved in 1966 that a naturally graded filiform Lie algebra is either isomorphic to L_n or Q_n , and where the last algebra exists only in even dimension. As L_n is clearly 1-abelian, a k -abelian filiform algebra with maximal abelianity index must be an isomorphic copy of Q_n . This forces the even dimensionality.

For $m \geq 4$ let $\mathfrak{g}_{(m,m-1)}^4$ be the Lie algebra whose structural equations are

$$\begin{aligned} d\omega_1 &= d\omega_2 = 0 \\ d\omega_j &= \omega_1 \wedge \omega_{j-1}, \quad 3 \leq j \leq 2m \\ d\omega_{2m+1} &= \sum_{j=2}^m (-1)^j \omega_j \wedge \omega_{2m+1-j} \end{aligned}$$

where $\{\omega_1, \dots, \omega_{2m+1}\}$ is a basis of $(\mathbb{C}^{2m+1})^*$.

Proposition. For any $m \geq 4$ the Lie algebra $\mathfrak{g}_{(m,m-1)}^4$ is naturally graded, $(m-1)$ -abelian of characteristic sequence $(2m-1, 1, 1)$.

Proof. We only observe that the $(m-1)$ -abelianity is given by the differential form $d\omega_{2m+1}$. \square

Notation. Recall that

$$\delta N_p^n = \{\mu \in N_p^n \mid \text{nilindex}(\mu) = p-1\}$$

Theorem. For $m \geq 4$ any naturally graded, central extension of L_{2m-1} by \mathbb{C} whose nilindex is $(2m-1)$ is isomorphic to $\mathfrak{g}_{(m,m-1)}^4$.

Proof. We analyze the elements of the set $\mathbf{E}_{c,1}^{t,2}(L_{2m-1})$ which have the prescribed nilindex.

For $t \neq m-1$ it is immediate to see that the nonexistence of such an extension. For $t = m-1$ the cocycles φ_{ij} defining the extension have to satisfy the relation

$$i + j = 2m + 1$$

The only ones which verify this requirement are the cocycles

$$\varphi_{j, 2m+1-j}, \quad 2 \leq j \leq \left\lfloor \frac{2m+1}{2} \right\rfloor$$

Moreover, if $\{X_1, \dots, X_{2m+1}\}$ is the dual basis of $\{\omega_1, \dots, \omega_{2m+1}\}$ we have

$$X_2 \wedge X_{2m-1} + (-1)^j X_j \wedge X_{2m+1-j} \in \Omega, \quad 3 \leq j \leq \left\lfloor \frac{2m+1}{2} \right\rfloor$$

Observe that the space $\frac{Ker\lambda}{\Omega}$ gives the presentation

$$\left\langle X_1, \dots, X_{2m+1} \mid X_2 \wedge X_{2m-1} + (-1)^j X_j \wedge X_{2m+1-j} = 0, \quad 3 \leq j \leq \left\lfloor \frac{2m+1}{2} \right\rfloor \right\rangle$$

and the underlying Lie algebra is obviously isomorphic to $\mathfrak{g}_{(m, m-1)}^4$ for any $m \geq 4$. \square

Corollary. For any values $(t, k_1, \dots, k_r) \neq (3, 2, 0, \dots, 0)$, $(r \geq 1)$ we have

$$\mathbf{E}_{c,1}^{t, k_1, \dots, k_r} \left(\mathfrak{g}_{(m, m-1)}^4 \right) = \emptyset$$

Remark. It is routine to verify that $\mathfrak{g}_{(m, m-1)}^4$ satisfies

$$S_q = T_q = \bigoplus_{k \geq q} \mathfrak{g}_k$$

where \mathfrak{g}_k denotes the k^{th} block of the associated graduation of $\mathfrak{g}_{(m, m-1)}^4$.

Let $\{X_1, \dots, X_{2m+1}\}$ denote the dual basis of $\{\omega_1, \dots, \omega_{2m+1}\}$ given for $\mathfrak{g}_{(m, m-1)}^4$.

Proposition. The linear mappings $ad(X_1), \dots, ad(X_{2m-1}), f_1^1, f_1^2, f_1^{2m+1}, f_2^2, f_2^{3+j} \ (1 \leq j \leq 2m-4), f_2^{2m}, f_2^{2m+1}$, where

$$\begin{cases} f_1^1(X_1) = X_1, \ f_1^1(X_2) = 0, \\ f_1^1(X_i) = (i-2)X_i, \ 3 \leq i \leq 2m, \\ f_1^1(X_{2m+1}) = (2m-3)X_{2m+1} \end{cases}$$

$$\begin{cases} f_2^2(X_1) = 0 \\ f_2^2(X_i) = X_i, \ 2 \leq i \leq 2m, \\ f_2^2(X_{2m+1}) = 2X_{2m+1} \end{cases}$$

$$\{f_1^2(X_1) = X_2, f_1^2(X_2) = X_{2m+1}\}$$

$$\{f_1^{2m+1}(X_1) = X_{2m+1}\}$$

$$\{f_2^{3+j}(X_k) = X_{k+j+1}, 1 \leq j \leq 2m-4, 2 \leq k \leq 2m-1-j\}$$

$$\{f_2^{2m}(X_2) = X_{2m}; f_2^{2m+1}(X_2) = X_{2m+1}\}$$

and the undefined images are zero, form a basis of $\text{Der}(\mathfrak{g}_{(m,m-1)}^4)$.

Proof. The listed endomorphisms are clearly derivations of $\mathfrak{g}_{(m,m-1)}^4$. Their linear independence is also easy to verify. As $\mathfrak{g}_{(m,m-1)}^4$ is naturally graded, a derivation d decomposes as

$$d = d_0 + d_1 + \dots + d_{2m-2}$$

where $d_j \in \text{Der}(\mathfrak{g}_{(m,m-1)}^4)$ and $d_j(\mathfrak{g}_k) \subset \mathfrak{g}_{k+j}$ for $1 \leq j \leq 2m-2$, $1 \leq k \leq 2m-1$. From this decomposition it is not difficult to prove that

$$\begin{aligned} d_0 &= a_0^1 f_1^1 + a_0^2 f_2^2 + a_0^3 f_1^2 \\ d_1 &= a_1^1 ad(X_1) + a_1^2 ad(X_2) \\ d_k &= a_k^1 ad(X_{k+1}) + a_k^2 f_2^{k+2}, \quad 2 \leq k \leq 2m-3 \\ d_{2m-2} &= a_{2m-2}^1 ad(X_{2m-1}) + a_{2m-2}^2 f_2^{2m} + a_{2m-2}^3 f_2^{2m+1} + a_{2m-2}^4 f_1^{2m+1} \end{aligned}$$

where $a_i^j \in \mathbb{C}$ for $0 \leq i \leq 2m-2$, $j = 1, 2, 3, 4$. □

Corollary. For $m \geq 4$, $\dim \text{Der}(\mathfrak{g}_{(m,m-1)}^4) = 4m + 1$.

Corollary. For $m \geq 4$, $\dim H^1(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4) = 2m$.

In particular, we see that for any $m \geq 4$ the rank of the Lie algebra $\mathfrak{g}_{(m,m-1)}^4$ is 2.

Let $H^2(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4)$ be the second Chevalley cohomology space. We consider cocycles φ which satisfy the following property :

1. For any $X \in \mathfrak{g}_{(m,m-1)}^4$ such that there exists $Y \in Z(\mathfrak{g}_{(m,m-1)}^4)$ with $Y \in \mathfrak{g}_{2m-1}$ and $Y \notin \text{im } ad(X)$ we have

$$\varphi(X, \mathfrak{g}_{(m,m-1)}^4) = 0$$

2. If $Y \in Z(\mathfrak{g}_{(m,m-1)}^4)$ is such that $Y \in \mathfrak{g}_{2m-1}$ then $Y \notin \text{im } (\varphi)$.

We say that the cocycle φ satisfies the property (P) and write

$$\widehat{Z}^2\left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4\right) = \left\{ \varphi \in Z^2\left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4\right) \mid \varphi \text{ satisfies (P)} \right\}$$

Then, for each k , we obtain the subspaces

$$\left\{ F_k \widehat{Z}^2\left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4\right) \right\}, \left\{ F_k \widehat{B}^2\left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4\right) \right\},$$

and

$$\left\{ F_k \widehat{H}^2\left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4\right) \right\}$$

For each $m \geq 4$ and $0 \leq r \leq 2m-4$, $2 \leq k \leq \lfloor \frac{2m-r}{2} \rfloor$ we define the cocycles

$$\psi_{k,r}(X_l, X_m) = \begin{cases} X_{k+j-1+r} & \text{if } l = k, m = j, 1+k \leq j \leq 2m+1-k-r \\ 0 & \text{otherwise} \end{cases}$$

Proposition. For any $m \geq 4$, the cocycles $\psi_{k,r}$ ($0 \leq r \leq 2m-4$, $2 \leq k \leq \lfloor \frac{2m-r}{2} \rfloor$) form a basis of $F_0 \widehat{Z}^2\left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4\right)$.

Moreover, for any $0 \leq j_0 \leq 2m-4$, a basis of $\widehat{Z}_{j_0}^2\left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4\right)$ is given by the cocycles ψ_{k,j_0} ($2 \leq k \leq \lfloor \frac{2m-j_0}{2} \rfloor$).

Proof. Let $\psi \in F_0 \widehat{Z}^2\left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4\right)$ be a cocycle. Then, for $2 \leq j \leq m-2$, $0 \leq h \leq 2m-j$, $1 \leq t \leq 2m-2j-h$ we have

$$\psi(X_j, X_{j+1+h}) = \alpha_{j,j+1+t+h}^{2j+t+h} X_{2j+t+h}$$

The cocycle condition $\delta(\psi) = 0$ implies the relations

$$\alpha_{j,j+1+h}^{2j+h} = \alpha_{j,j+1+t+h}^{2j+t+h}$$

for $2 \leq j \leq m-2$, $0 \leq h \leq 2m-j$, $1 \leq t \leq 2m-2j-h$, and

$$\begin{aligned} \alpha_{m-1,m}^{2m-2} &= \alpha_{m-1,m+1}^{2m-1} \\ \alpha_{m-1,m}^{2m-1} &= \alpha_{m-1,m+1}^{2m} \\ \alpha_{m,m+1}^{2m} &= \alpha_{m-1,m}^{2m-2} - \alpha_{m-1,m+2}^{2m} \end{aligned}$$

Thus the cocycle ψ can be rewritten as :

$$\begin{aligned} \psi &= \sum_{j=2}^{m-2} \left(\alpha_{j,j+1}^{2j} \psi_{j,0} + \dots + \alpha_{j,j+1}^{2m} \psi_{j,2m+1-j} \right) + \alpha_{m-1,m}^{2m-2} \psi_{m-1,0} + \alpha_{m-1,m}^{2m-1} \psi_{m-1,m} + \\ &+ \alpha_{m-1,m}^{2m} \psi_{m-1,2} + \alpha_{m,m+1}^{2m} \psi_{m,0} \end{aligned}$$

So the cocycles $\psi_{k,r}$ ($0 \leq r \leq 2m-4$, $2 \leq k \leq \lfloor \frac{2m-r}{2} \rfloor$) constitute a generator system for $F_0 \widehat{Z}^2\left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4\right)$. Clearly they are linearly independent, so they form a basis. This proves the first assertion. The second follows immediately. \square

Corollary. For $m \geq 4$ and $0 \leq t \leq m-3$,

$$\dim \widehat{Z}_0^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right) = m-1$$

$$\dim \widehat{Z}_{2t+1}^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right) = \dim \widehat{Z}_{2t+2}^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right) = m-t-2$$

In particular

$$\dim F_0 \widehat{Z}^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right) = m^2 - 2m + 1$$

Proposition. For any $m \geq 4$, the cohomology class $[\psi_{k,r}]$ of the cocycles $\psi_{k,r}$ ($1 \leq r \leq 2m-4$) are nontrivial and linearly independent.

Moreover, $F_0 \widehat{H}^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right)$ is a $(m^2 - 3m + 2)$ -dimensional subspace of $F_0 H^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right)$.

Proof. From the basis obtained in the previous proposition it follows easily that $F_0 \widehat{B}^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right) = \{\psi_{k,0} \mid 2 \leq k \leq m\}$. Observe in particular that most coboundaries have been previously excluded from the cohomology space $F_0 \widehat{Z}^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right)$ by property (P). \square

Corollary. For any $1 \leq r \leq 2m-4$ the cocycles $\psi_{k,r}$ ($2 \leq k \leq \lfloor \frac{2m-r}{2} \rfloor$) form a basis of $\widehat{H}^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right)$.

Proof. Follows from the previous result, as for $r \geq 1$

$$\widehat{H}_r^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right) = \widehat{Z}_r^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right)$$

\square

Proposition. For $k \geq 1$, any cocycle $\psi_k \in \widehat{H}_k^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right)$ which satisfies $\psi_k \left(C^1 \mathfrak{g}_{(m,m-1)}^4, C^1 \mathfrak{g}_{(m,m-1)}^4 \right) = 0$ is linearly expandable if and only if $\psi_k = \lambda \psi_{2,k}$, $\lambda \in C \setminus \{0\}$.

Moreover $\mathfrak{g}_{(m,m-1)}^4 + \lambda \psi_{2,k} \simeq \mathfrak{g}_{(m,m-1)}^4 + \psi_{2,k}$ for any $\lambda \neq 0$.

Theorem. Let $\psi_k \in \widehat{H}_k^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right)$ by a linearly expandable cocycle which satisfies $\psi_k \left(C^1 \mathfrak{g}_{(m,m-1)}^4, C^1 \mathfrak{g}_{(m,m-1)}^4 \right) = 0$. Then the deformation $\mathfrak{g}_{(m,m-1)}^4 + \psi_k$ is isomorphic to the nilradical of a solvable, complete Lie algebra $\mathfrak{r}_{(m,m-1)}^{4,k}$ of dimension $2m+2$. Moreover, if $k = 2m-5$ or $k = 2m-4$, then the Lie algebra $\mathfrak{r}_{(m,m-1)}^{4,k}$ is rigid.

Proof. We have seen that the rank of the algebra $\mathfrak{g}_{(m,m-1)}^4$ is 2 for any $m \geq 4$. From the previous proposition we know that $\psi_k = \lambda \psi_{2,k}$ for $\lambda \neq 0$. Without loss of generality we can take $\lambda = 1$. Let $f \in \text{Der}(\mathfrak{g}_{(m,m-1)}^4 + \psi_{2,k})$ such that

$$f(X_t) = \sum_{j=1}^{2m+2} f_t^j X_j, \quad 1 \leq t \leq 2m+2$$

The cocycle $\psi_{2,k}$ implies in particular

$$\psi_{2,k}(X_2, X_j) = X_{1+j+k}, \quad 3 \leq j \leq 2m-1-k$$

Taking the relation $[f(X_2), X_3] + [X_2, f(X_3)] = f(X_{4+k})$, we obtain

$$f_2^2 = (1+k) f_1^1$$

This proves that the deformation $\mathfrak{g}_{(m,m-1)}^4 + \psi_{2,k}$ has rank one. Now let $\mathfrak{r}_{(m,m-1)}^{4,k}$ be the solvable Lie algebra whose Cartan-Maurer equations on the basis $\{\omega_1, \dots, \omega_{2m+1}, \theta\}$ are

$$\begin{aligned} d\omega_1 &= \theta \wedge \omega_1 \\ d\omega_2 &= (k+1)\theta \wedge \omega_2 \\ d\omega_j &= \omega_1 \wedge \omega_{j-1} + (k+j-1)\theta \wedge \omega_j, \quad 3 \leq j \leq 3+k \\ d\omega_{4+k+j} &= \omega_1 \wedge \omega_{3+k+j} + \omega_2 \wedge \omega_{3+j} + (k+5+j)\theta \wedge \omega_{4+k+j}, \quad 0 \leq j \leq 2m-4-k \\ d\omega_{2m+1} &= \sum_{j=2}^m (-1)\omega_j \wedge \omega_{2m+1-j} + (2k+2m-1)\theta \wedge \omega_{2m+1} \\ d\theta &= 0 \end{aligned}$$

It is immediate to verify that this Lie algebra is solvable and complete. If $k = 2m-5$ or $2m-4$ the algebra is moreover (see [1] for a standard proof). As we have $\mathfrak{r}_{(m,m-1)}^{4,k} \simeq (\mathfrak{g}_{(m,m-1)}^4 + \psi_{2,k}) \oplus \mathfrak{t}$, its nilradical is clearly isomorphic to $\mathfrak{g}_{(m,m-1)}^4 + \psi_{2,k}$. \square

Remark. The previous proposition provides us with a correspondence ($k = 2m-4$, $k = 2m-5$)

$$I : \widehat{H}^2(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4) \rightarrow \mathbb{Z}^{2m+1}$$

between the cohomology class of the cocycle $\psi_{2,k}$ and the $(2m+1)$ -uples $(1, k+1, \dots, 2k+2m-1)$, where this sequence gives the weight distribution of the eigenvalues of the action of the torus \mathfrak{t} over $\mathfrak{g}_{(m,m-1)}^4 + \psi_{2,k}$.

Let $\psi \in F_1 \widehat{H}^2(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4)$ be a cocycle. Then this cocycle is expressible as

$$\psi = \psi_1 + \dots + \psi_r, \quad \psi_i \in \widehat{H}_i^2(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4)$$

If r is the greatest integer such that $\psi_r \neq 0$, then we call ψ_r the sill cocycle of ψ .

Lemma. *For any cocycle $\psi \in F_1 \hat{H}^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right)$ the sill cocycle ψ_r ($1 \leq r \leq 2m-4$) is linearly expandable.*

Remark. *This fact is already known for the deformations of the filiform Lie algebra L_n [21]. For this model the property is preserved, as it is a central extension of the last algebra. The Lie algebra $\mathfrak{g}_{(m,m-1)}^4 + \psi_r$, where ψ_r is the sill cocycle, is usually called sill algebra.*

Theorem. *Let $\psi \in F_1 \hat{H}^2 \left(\mathfrak{g}_{(m,m-1)}^4, \mathfrak{g}_{(m,m-1)}^4 \right)$ be a linearly expandable cocycle such that the sill cocycle ψ_r satisfies*

1) $r \leq 2m-5$

2) $\psi_r \left(C^1 \mathfrak{g}_{(m,m-1)}^4, C^1 \mathfrak{g}_{(m,m-1)}^4 \right) = 0$.

If there exists a component ψ_{j_0} of ψ such that $j_0 \leq r-1$ and satisfying $\psi_{j_0} \left(C^1 \mathfrak{g}_{(m,m-1)}^4, C^1 \mathfrak{g}_{(m,m-1)}^4 \right) \neq 0$, then the Lie algebra $\mathfrak{g}_{(m,m-1)}^4 + \psi$ is characteristically nilpotent.

Proof. As the sill cocycle is itself linearly expandable, we know from previous results that the algebra $\left(\mathfrak{g}_{(m,m-1)}^4 + \psi_r \right)$ has rank one. If $f \in \text{Der} \left(\mathfrak{g}_{(m,m-1)}^4 + \psi \right)$, it is easy to verify that the entries of the matrix $(f_j^i)_{1 \leq i, j \leq 2m+1}$ of f satisfy the relations

$$\begin{aligned} f_j^i &= 0, \quad 2 \leq i < j \leq 2m+1 \\ f_i^i &\in \langle f_1^1, f_2^2 \rangle_{\mathbb{C}}, \quad 3 \leq j \leq 2m+1 \end{aligned}$$

Thus the sill cocycle implies

$$f_2^2 = (r+1) f_1^1$$

Now let ψ_{j_0} the summand of ψ that satisfies $\psi_{j_0} \left(C^1 \mathfrak{g}_{(m,m-1)}^4, C^1 \mathfrak{g}_{(m,m-1)}^4 \right) \neq 0$. Then there exists an $s \geq 4$ such that

$$\psi_{j_0} (X_3, X_s) = \lambda_{3s} X_{1+s+j_0}$$

From this equation we obtain

$$(s-1) f_1^1 + 2f_2^2 = (1+s+j_0-2) f_1^1 + f_2^2$$

so that

$$f_2^2 = j_0 f_1^1$$

and as $j_0 \leq r-1$ we have $f_1^1 = f_2^2 = 0$. It is not difficult to see that the other summands of ψ do not affect these relations. It follows that the matrix of f is strictly upper triangular, thus the derivation f is nilpotent and $\left(\mathfrak{g}_{(m,m-1)}^4 + \psi \right)$ is characteristically nilpotent. \square

Corollary. For any $m \geq 4$ and $k \geq 3$, $1 \leq s \leq r-1 \leq 2m-6$ the Lie algebra

$$\mathfrak{g}_{(m,m-1)}^4 + \alpha\psi_{2,r} + \beta\psi_{k,s}, \quad \alpha, \beta \in \mathbb{C} - \{0\}$$

is characteristically nilpotent.

4 Extensions of $\mathfrak{g}_{(m,m-1)}^4$

In this last section we study how the previous deformations are compatible with central extensions of $\mathfrak{g}_{(m,m-1)}^4$ by \mathbb{C} . We will obtain, for any $m \geq 4$, characteristically nilpotent Lie algebras of characteristic sequence $(2m-1, 2, 1)$ in dimension $2m+2$.

For $m \geq 4$ let us define the Lie algebras $\mathfrak{g}_{(m,m-1)}^{4,1}$ whose Cartan-Maurer equations are

$$\begin{aligned} d\omega_1 &= d\omega_2 = 0 \\ d\omega_j &= \omega_1 \wedge \omega_{j-1}, \quad 3 \leq j \leq 2m \\ d\omega_{2m+1} &= \sum_{j=2}^m (-1)^j \omega_j \wedge \omega_{2m+1-j} \\ d\omega_{2m+2} &= \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^m (-1)^j (m+1-j) \omega_j \wedge \omega_{2m+2-j} \end{aligned}$$

Remark. It is clear that these algebras have characteristic sequence $(2m-1, 2, 1)$ for any $m \geq 4$ and that they are $(m-1)$ -abelian. The only property which got lost by the extension is the natural graduation.

Moreover, let $\bar{\psi}_{2,k}$ be the prolongation by zeros of $\psi_{2,k}$, i.e.,

$$\bar{\psi}_{2,k} = \begin{cases} \psi_{2,k}(X_l, X_m) & \text{if } 1 \leq l, m \leq 2m+1 \\ 0 & \text{if } l = 2m+2 \text{ or } m = 2m+2 \end{cases}$$

Proposition. The cocycle $\bar{\psi}_{2,k} \in H^2(\mathfrak{g}_{(m,m-1)}^{4,1}, \mathfrak{g}_{(m,m-1)}^{4,1})$ is linearly expandable if and only if $k = 2m-5$ or $k = 2m-4$.

Proof. The only manner in which the differential form $d\omega_{2m+2}$ is closed is that the cocycle $\bar{\psi}_{2,k}$ only affects the differential form $d\omega_{2m}$ adding the exterior product $\omega_2 \wedge \omega_3$ or changing the differential forms $d\omega_{2m-1}, d\omega_{2m}$ by adding, respectively, the exterior products $\omega_2 \wedge \omega_3$ and $\omega_2 \wedge \omega_4$. This corresponds to the indexes $k = 2m-4$ and $k = 2m-5$. \square

Remark. Observe that the prolongation by zeros of a cocycle will be linearly expandable in the extension if it is compatible with the adjoined differential form $d\omega_{2m+2}$. This result can be easily generalized for prolongations of cocycles $\psi_{t,k}$ for arbitrary $t \geq 3$, but we are only considering the cocycles for which $t = 2$.

The two next theorems are direct consequences of the results of the previous sections :

Theorem. For $m \geq 4$ let $\mathfrak{r}_{(m,m-1)}^{4,1,2m-5}$ be the Lie algebra whose Cartan-Maurer equations are

$$\begin{aligned}
d\omega_1 &= \theta \wedge \omega_1 \\
d\omega_2 &= (2m-4)\theta \wedge \omega_2 \\
d\omega_j &= \omega_1 \wedge \omega_{j-1} + (2m-6+j)\theta \wedge \omega_j, \quad 3 \leq j \leq 2m-2 \\
d\omega_{2m-1} &= \omega_1 \wedge \omega_{2m-2} + \omega_2 \wedge \omega_3 + (4m-7)\theta \wedge \omega_{2m-1} \\
d\omega_{2m} &= \omega_1 \wedge \omega_{2m-1} + \omega_2 \wedge \omega_4 + (4m-6)\theta \wedge \omega_{2m} \\
d\omega_{2m+1} &= \sum_{j=2}^m (-1)^j \omega_j \wedge \omega_{2m+1-j} + (6m-11)\theta \wedge \omega_{2m+1} \\
d\omega_{2m+2} &= \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^m (-1)^j (m+1-j) \omega_j \wedge \omega_{2m+2-j} + (6m-10)\theta \wedge \omega_{2m+2} \\
d\theta &= 0
\end{aligned}$$

This algebra is solvable, decomposable and rigid. Moreover, its nilradical is isomorphic to $\mathfrak{g}_{(m,m-1)}^{4,1} + \bar{\psi}_{2,2m-5}$.

Theorem. For $m \geq 4$ the Lie algebra $e_1 \left(\mathfrak{g}_{(m,m-1)}^4 + \psi_{2,2m-4} \right)$ defined by

$$\begin{aligned}
d\omega_1 &= d\omega_2 = 0 \\
d\omega_j &= \omega_1 \wedge \omega_{j-1}, \quad 3 \leq j \leq 2m-1 \\
d\omega_{2m} &= \omega_1 \wedge \omega_{2m-1} + \omega_2 \wedge \omega_3 \\
d\omega_{2m+1} &= \sum_{j=2}^m (-1)^j \omega_j \wedge \omega_{2m+1-j} \\
d\omega_{2m+2} &= \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^m (-1)^j (m+1-j) \omega_j \wedge \omega_{2m+2-j}
\end{aligned}$$

is characteristically nilpotent. Moreover, its characteristic sequence is $(2m-1, 2, 1)$.

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